Control of underactuated systems – from theory to practice *

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1. Mathematical Properties of the Model

2. Robot Model

3. Largest feedback linearizable subsystem

4. Stabilization problem

5. Simulation Results
Overview

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Mathematical Properties of the Model

Consider the system with multiple inputs with dynamics given by:

\[
\dot{x} = f(x) + G(x)u,
\]  

(1)

where

\[
\begin{align*}
  f(x) &= \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \\
  G(x) &= \begin{bmatrix} g_{11}(x) & \cdots & g_{1m}(x) \\ \vdots & \ddots & \vdots \\ g_{n1}(x) & \cdots & g_{nm}(x) \end{bmatrix} = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix}. 
\end{align*}
\]

(2)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( f(x), G(x) \) are of appropriate dimensions.

We can define the distributions

\[
D_j = \text{span}\{g_1, \ldots, g_m, \text{ad}_f g_1, \ldots, \text{ad}_f g_m, \ldots, \text{ad}_f^{j-1} g_1, \ldots, \text{ad}_f^{j-1} g_m\}
\]

where:

\[
\text{ad}_f g_i = [f, g_i] = \frac{\partial g_i}{\partial x} f - \frac{\partial f}{\partial x} g_i, \text{ for any } k \geq 1, \text{ setting } \text{ad}_f^0 g_i(x) = g_i(x),
\]

and let \( \bar{D}_j \) denote the involutive closure of \( D_j \), which is the smallest involutive distribution containing \( D_j \) and \( j = 0, 1, \ldots, n - 1 \).

* Distribution \( D \) in involutive if the Lie Bracket \([f_i(x), f_j(x)]\) for any pair of vector fields \( f_i(x), f_j(x) \) belonging to \( D \) is a vector field which belongs to \( D \).
Mathematical Properties of the Model

Aim

1. analyze properties of underactuated 3 DOF pendulum
2. stabilize it in vertical position

- When the system is underactuated, full feedback linearisation is not possible.
- The system should be decomposed into two subsystems, one which is linear, and one which stays still nonlinear.
- An important issue is the maximal dimension of the linear subsystem that might be obtained.
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Robot Model

3-link robot
- \( N = 3 \) rigid bodies coupled in a tree structure
- supported on ground via an actuated frictionless revolute joint
- one degree of underactuation (3 DOF with 2 independent actuators)

Table 1: Robot parameters

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Robot Model

In order to establish the system dynamics one can define Lagrangian

\[ L = K - V \]

while \( K = \frac{1}{2} \dot{q}^T M(q) \dot{q} \) denotes kinetic energy, with \( M \) being a positive definite inertia matrix, and \( V \) is the potential energy.

Next, taking into account the actuation on the system (Fig. 2) one obtains

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \begin{cases} 
\tau_k, & k = 1, 2 \\
0, & k = 3
\end{cases}
\]

(3)

with \( \tau_k \in \mathbb{R} \).

Figure 2: Triple pendulum – underactuated model
The overall model of dynamics can be written in a standard form of:

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \]  

(4)

where matrices \( M, C, G \) are as following:

\[ M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \]  

(5)

or in equivalent form:

\[
\begin{align*}
m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 + \mu_1 + G_1 &= \tau_1 \\
m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 + \mu_2 + G_2 &= \tau_2 \\
m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 + \mu_3 + G_3 &= 0
\end{align*}
\]  

(6)

where:

\[
\begin{align*}
\mu_1 &= c_{11}\dot{q}_1 + c_{12}\dot{q}_2 + c_{13}\dot{q}_3, \\
\mu_2 &= c_{21}\dot{q}_1 + c_{22}\dot{q}_2 + c_{23}\dot{q}_3, \\
\mu_3 &= c_{31}\dot{q}_1 + c_{32}\dot{q}_2 + c_{33}\dot{q}_3.
\end{align*}
\]
The elements of the $M$ mass matrix are as follows:

\begin{align*}
    m_{11} &= a_1 + a_2 + a_3 + a_4 + a_5 + 2(r_1 + r_2 + r_3) \\
    m_{12} &= a_2 + a_3 + a_4 + r_1 + r_2 + 2r_3 \\
    m_{13} &= a_3 + r_1 + r_3 \\
    m_{21} &= m_{12} \\
    m_{22} &= a_2 + a_3 + a_4 + 2r_3 \\
    m_{23} &= a_3 + r_3 \\
    m_{31} &= m_{13} \\
    m_{32} &= m_{23} \\
    m_{33} &= a_3
\end{align*}

(7)

where

\begin{align*}
    a_1 &= m_1 L_{c_1}^2 + I_1 \\
    a_2 &= m_2 L_{c_2}^2 + I_2 \\
    a_3 &= m_3 L_{c_3}^2 + I_3 \\
    a_4 &= m_3 L_2^2 \\
    a_5 &= (m_2 + m_3)L_1^2 \\
    r_1 &= L_1 L_{c_3} m_3 \cos(q_2 + q_3) \\
    r_2 &= L_1 (L_2 m_3 + L_{c_2} m_2) \cos q_2 \\
    r_3 &= L_2 L_{c_3} m_3 \cos q_3.
\end{align*}

(8)
Matrix $C$ and $G$

The Coriolis matrix $C$ is:

\[
\begin{align*}
    c_{11} &= -d_1 \dot{q}_2 - d_2 \dot{q}_3 \\
    c_{12} &= -d_1(\dot{q}_1 + \dot{q}_2) - d_2 \dot{q}_3 \\
    c_{13} &= -d_2(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\
    c_{21} &= d_1 \dot{q}_1 - d_3 \dot{q}_3 \\
    c_{22} &= -d_3 \dot{q}_3 \\
    c_{23} &= -d_3(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\
    c_{31} &= d_2 \dot{q}_1 + d_3 \dot{q}_2 \\
    c_{32} &= d_3(\dot{q}_1 + \dot{q}_2) \\
    c_{33} &= 0
\end{align*}
\]

(9)

with

\[
\begin{align*}
    d_1 &= L_1 L_c m_3 \sin(q_2 + q_3) + (m_2 L_c + m_3 L_2) L_1 \sin q_2 \\
    d_2 &= L_1 L_c m_3 \sin(q_2 + q_3) + L_2 L_c m_3 \sin q_3 \\
    d_3 &= L_2 L_c m_3 \sin q_3.
\end{align*}
\]

(10)

The Gravity force matrix $G$ is as follows:

\[
\begin{align*}
    G_1 &= g(b_1 + b_2 + b_3) \\
    G_2 &= g(b_2 + b_3) \\
    G_3 &= g b_3
\end{align*}
\]

(11)

where:

\[
\begin{align*}
    b_1 &= m_1 L_c \cos q_1 + (m_2 + m_3) L_1 \cos q_1 \\
    b_2 &= (m_2 L_c + m_3 L_2) \cos(q_1 + q_2) \\
    b_3 &= m_3 L_c \cos(q_1 + q_2 + q_3). \\
    g &= \text{gravitational acceleration}
\end{align*}
\]

(12)
Let’s recall the equations of motion in the following form

\[
\begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{12} & m_{22} & m_{23} \\
m_{13} & m_{23} & m_{33}
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2 \\
\ddot{q}_3
\end{bmatrix}
+ \begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3
\end{bmatrix}
+ \begin{bmatrix}
G_1 \\
G_2 \\
G_3
\end{bmatrix}
= \begin{bmatrix}
\tau_1 \\
\tau_2 \\
0
\end{bmatrix}.
\]

(13)

and assume that \( C_1 = [c_{11}, c_{12}, c_{13}] \dot{q} \), \( C_2 = [c_{21}, c_{22}, c_{23}] \dot{q} \), \( C_3 = [c_{31}, c_{32}, c_{33}] \dot{q} \).

In the following step, we can linearize this dynamics with the use of collocated linearization

\[
\ddot{q}_3 = -\frac{m_{13}\ddot{q}_1 + m_{23}\ddot{q}_2 + C_3 + G_3}{m_{33}}
\]

Introduce linearizing controller:

\[
\tau_1 = \overline{m}_{11}v_1 + \overline{m}_{12}v_2 + \overline{C}_1 + \overline{G}_1
\]
\[
\tau_2 = \overline{m}_{21}v_1 + \overline{m}_{22}v_2 + \overline{C}_2 + \overline{G}_2
\]

(14)
Partial linearization conditions

where
\[
\begin{align*}
\bar{m}_{11} &= m_{11} + J_1 m_{31} & \bar{C}_1 &= C_1 + J_1 \mu_3 \\
\bar{m}_{12} &= m_{12} + J_1 m_{32} & \bar{C}_2 &= C_2 + J_2 \mu_3 \\
\bar{m}_{21} &= m_{21} + J_2 m_{31} & \bar{G}_1 &= G_1 + J_1 G_3 \\
\bar{m}_{22} &= m_{22} + J_2 m_{32} & \bar{G}_2 &= G_2 + J_2 G_3.
\end{align*}
\]

for \( J_1 = -\frac{m_{13}}{m_{33}} \), \( J_2 = -\frac{m_{23}}{m_{33}} \).

and \( v_1 \) i \( v_2 \) are additional control inputs, described later.

Calculations are valid when the system is not in its singularity, when:

1. \( \det \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{21} & \bar{m}_{22} \end{bmatrix}^{-1} = \frac{m_{33}}{\det M} \neq 0 \),

Here \( m_{33} > 0 \) and \( \det M > 0 \) by definition.

2. \( J_1 \neq 0 \) and \( J_2 \neq 0 \), respectively, for two cases:
   - \( a_3 = m_3 L_3 (L_1 + L_2) \) for \( q_2 = 0, q_3 = \pi + 2k\pi \);
     \( a_3 < m_3 L_3 (L_1 + L_2) \) for solution of the following equation: \( a_3 = -r_1 - r_3 \).
   - \( a_3 = m_3 L_2 L_3 \) for \( q_3 = \pi + 2k\pi \);
     \( a_3 < m_3 L_2 L_3 \) for \( q_3 = -\arccos(\frac{a_3}{m_3 L_2 L_3}) \).
Partial linearization conditions

In Eq. (14) variables $v_1$ and $v_2$ are new control inputs. Thus, considered system can be written in the following form

$$
\ddot{q}_1 = v_1 \\
\ddot{q}_2 = v_2 \\
\ddot{q}_3 = -m_{33}^{-1} (m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + C_3 + G_3)
$$

or alternatively, introducing the state vector as:

$$
x = [q_1, w_1, q_2, w_2, q_3, w_3]^\top
$$

and substituting $C_3 + G_3 = R_3 - J_1R_1 - J_2R_2$, the pendulum model is

$$
\dot{q}_1 = w_1 \\
\dot{w}_1 = v_1 \\
\dot{q}_2 = w_2 \\
\dot{w}_2 = v_2 \\
\dot{q}_3 = w_3 \\
\dot{w}_3 = R_3 + J_1(v_1 - R_1) + J_2(v_2 - R_2).
$$
Using more general form, the above equation (17) can be written as:

\[ \dot{x} = f(x) + g(x)u \]

or

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{w}_1 \\
\dot{q}_2 \\
\dot{w}_2 \\
\dot{q}_3 \\
\dot{w}_3
\end{bmatrix} =
\begin{bmatrix}
w_1 \\
0 \\
w_2 \\
0 \\
w_3 \\
R_3 - J_1 R_1 - J_2 R_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
J_1
\end{bmatrix} v_1 +
\begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
J_2
\end{bmatrix} v_2. \tag{18}
\]

where: \( J_1(q_2, q_3) = -\frac{m_{13}(q_2, q_3)}{m_{33}} \), \( J_2(q_3) = -\frac{m_{23}(q_3)}{m_{33}} \) and \( R_i = M^{-1}(i)(-C(q, \dot{q}) \dot{q} - G) \), where \( M^{-1}(i) \) is an \( i \)-th row of the inverse of Mass matrix \( M \).
Overview

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5. Simulation Results
As mentioned before, underactuated systems are not fully linearizable. The question arises – what is the largest feedback linearizable subsystem of the whole system?

In order to find the largest linearizable subsystem we propose to analyze the following distributions:

- \( D_0 = \text{span}\{g_1, g_2\} \) – obviously is involutive
- \( D_1 = \text{span}\{g_1, g_2, [f, g_1], [f, g_2]\} \) – not involutive

One needs to find smallest involutive closure of \( D_1 \):

- \( \overline{D}_1 = \text{span}\{g_1, g_2, [f, g_1], [f, g_2], [g_1, ad_f g_1]\} \) – not involutive
- \( \overline{D}_1 = \text{span}\{g_1, g_2, [f, g_1], [f, g_2], [g_2, ad_f g_2]\} \) – not involutive
- other combinations – not involutive
- \( \overline{D}_1 = \text{span}\{g_1, g_2, [f, g_1], [f, g_2], [g_1, ad_f g_2], [ad_f g_1, ad_f g_2]\} \) – involutive
Largest feedback linearizable subsystem

Frobenius Theorem

A nosingular distribution is completely integrable if and only if is involutive.

Then one needs to find an output function $h$ that anihilates $\overline{D}_1$, i.e.

$$\begin{bmatrix}
\frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \frac{\partial h}{\partial x_4} & \frac{\partial h}{\partial x_5} & \frac{\partial h}{\partial x_6}
\end{bmatrix}
\begin{bmatrix}
g_1 & g_2 & [f, g_1] & [f, g_2] & [g_1, ad_f g_1] & [ad_f g_1, ad_f g_2]
\end{bmatrix} = 0$$

As a result we get:

$$\begin{align*}
\frac{\partial h}{\partial w_1} &= 0, \\
\frac{\partial h}{\partial w_2} &= 0, \\
\frac{\partial h}{\partial q_1} &= 0, \\
\frac{\partial h}{\partial q_2} &= 0, \\
\frac{\partial h}{\partial w_3} &= 0, \\
\frac{\partial h}{\partial q_3} &= 0.
\end{align*}$$

(19)

It is trivial that the only solution of Eq (19) is $h = \text{constant}$ because $\overline{D}_1$ is of full rank 6.

As a conclusion – the largest feedback linearizable subsystem is of dimension 4.
The Lie brackets used in the above calculations are as follows:

\[
\begin{align*}
[f, g_1] &= \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^\top - J_1 F_{16}^\top \\
[f, g_2] &= \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^\top - J_2 F_{26}^\top \\
[g_1, ad_f g_1] &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top F_{56}^\top \\
[g_1, ad_f g_2] &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top F_{66}^\top \\
[g_2, ad_f g_1] &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top F_{76}^\top \\
[g_2, ad_f g_2] &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top F_{86}^\top \\
[ad_f g_1, ad_f g_2] &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top (F_{95} F_{96})^\top ,
\end{align*}
\]  

where:

\[
\begin{align*}
g_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top J_1 \\
g_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^\top J_2 \\
F_{16} &= \frac{1}{L_3^3} (L_1 \sin(q_2 + q_3) [2 w_1 + w_2 + w_3] + L_2 \sin(q_3) [2 w_1 + 2 w_2 + w_3]) \\
F_{26} &= \frac{1}{L_3^3} (L_2 \sin(q_3) (2 w_1 + 2 w_2 + w_3)) \\
F_{56} &= -\frac{1}{L_3^2} ((\sin(2q_2 + 2q_3)L_1^2 + 2 \sin(q_2 + 2q_3) L_1 L_2 + \sin(2q_3)L_2^2)) \\
F_{66} &= -\frac{1}{L_3^2} (L_1 L_2 \sin(q_2 + 2q_3) + L_2^2 \sin(2q_3)) \\
F_{76} &= -\frac{1}{L_3^2} (L_1 L_2 \sin(q_2 + 2q_3) + L_2^2 \sin(2q_3)) \\
F_{86} &= -\frac{1}{L_3^2} L_2^2 \sin(2q_3) \\
F_{95} &= \frac{1}{L_3^2} L_1 L_2 \sin(q_2) \\
F_{96} &= \frac{1}{L_3^2} L_1 L_2 w_2 \cos(q_2 + 2q_3)
\end{align*}
\]
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Aim

examine an implementation of a hybrid controller to stabilize a triple pendulum around its top unstable position, taking into account the limitations and constraints resulting from practical conditions (existing robot)

Stabilization will be obtained with the two commonly known approaches

- first – which utilizes the collocated methods for linearization
- second – the additional LQR controller is used to stabilize the system near the equilibrium point.
Stabilizing controller

\[ u = \begin{cases} 
  u_h & \text{for swing,} \\
  u_{Lin} & \text{for stabilization.} 
\end{cases} \tag{21} \]

\( u_h \) — is used to bring the pendulum near the equilibrium pose,

\[ u_h = [\tau_1, \tau_2]^\top \tag{22} \]

\( u_{Lin} \) — to stabilize at equilibrium

\[ u_{Lin} = -K(x_r - x). \tag{23} \]

\[ x_r = [q^d_1, q^d_2, q^d_3, \dot{q}^d_1, \dot{q}^d_2, \dot{q}^d_3]^\top \] and \( K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\
 k_7 & k_8 & k_9 & k_{10} & k_{11} & k_{12} \end{bmatrix} \), stand for the reference state and the controller gains, respectively.

\( \tau_1, \tau_2 \) are given by Eq. (14) and

\[
\begin{align*}
v_1 &= \ddot{q}_1 = \ddot{q}^d_1 + K_1^D (\dot{q}^d_1 - \dot{q}_1) + K_1^P (q^d_1 - q_1) \\
v_2 &= \ddot{q}_2 = \ddot{q}^d_2 + K_2^D (\dot{q}^d_2 - \dot{q}_2) + K_2^P (q^d_2 - q_2)
\end{align*} \tag{24} \tag{25}
\]

where \( K_1^D, K_1^P, K_2^D \) i \( K_2^P \) are positive gains, and \( q^d_1, q^d_2, q^d_3, \dot{q}^d_1, \dot{q}^d_2, \dot{q}^d_3 \) denote desired values at the equilibrium point.
Zero Dynamics

The zero dynamics was obtained assuming that $h = \text{const}$ and

$$q_1^d = \frac{\pi}{2}, \quad q_2^d = 0, \quad \dot{q}_1^d = 0, \quad \dot{q}_2^d = 0, \quad \ddot{q}_1^d = 0, \quad \ddot{q}_2^d = 0.$$ 

The resulting zero dynamics is calculated as follows

$$\ddot{q}_3 = \xi \sin q_3$$  \hspace{1cm} (26)

where: $\xi = \frac{1}{a_3}gm_3L_3$, and partial solution of Eq. (26) is given by Eq. (27), for some constant $e_1$:

$$\dot{q}_3 = -\sqrt{2\xi} \cos q_3 + e_1$$  \hspace{1cm} (27)

Zero dynamics phase portrait (Fig. 3) was obtained numerically, is locally stable and formed by closed curves.

Figure 3: Zero dynamics
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Simulation Results

Existing robot being investigated in simulations

3-link robot
- driven by Maxon 200W EC-Powermax 30 brushless motors
- planetary gearhead of $N = 53$
- maximum torque of approximately 6 Nm

Table 2: Robot parameters

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Figure 4: 3-link pendulum – experimental test-bed
Simulation Results

Simulation conditions

- the desired stabilization pose is the upright position for which the angles $q_1^d$, $q_2^d$ and $q_3^d$ were equal $90^\circ$, $0$ and $0$, respectively.
- initial condition: $q_{10} = 20^\circ$, $q_{20} = -60^\circ$ and $q_{30} = 131^\circ$  (exemplary one)
- the torque magnitude is restricted to 6 Nm – taken from existing robot
- simulation time $t = 10$ s.

The obtained angular trajectories are shown in Fig. 5a, while the control signal produced by motor is depicted on Fig. 5b.

![Figure 5: a) Angular position of links, b) Motor torque c) Animation.](image)
Thank You
For Your Attention